Math 131B-1: Homework 3

Due: January 24, 2014

1. Read Apostol Sections 4.2-4, 2.12-13, 3.8-12.
2. Do problems 4.8, 4.9, 2.18, 2.19, 3.16, and 3.19 in Apostol.
3. Let (M, d') be any infinite set M with the discrete metric (i.e. $d'(x, y) = 1$ when $x \neq y$).
   - Show that (M, d') is complete.
   - Let S be any infinite subset of M. Show that S is closed and bounded, but not compact.
4. *Two metrics with the same convergence properties.* Consider the space $\mathbb{R}^n$ with two metrics d_1 and d_2 defined as follows. If $\mathbf{x} = (x_1, \cdots x_n)$ and $\mathbf{y} = (y_1, \cdots y_n)$, we let

$$d_1(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$
$$d_2(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : 1 \leq i \leq n\}$$
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   That is, d_1 is the usual metric and d_2 is the "box metric."

   - Let $\mathbf{x} \in \mathbb{R}^n$. Show that any ball $B_{(\mathbb{R}^n, d_1)}(\mathbf{x}; r)$ contains a ball $B_{(\mathbb{R}^n, d_2)}(\mathbf{x}; r')$ for some $r' \leq r$. Likewise show that any ball $B_{(\mathbb{R}^n, d_2)}(\mathbf{x}; r)$ contains a ball $B_{(\mathbb{R}^n, d_1)}(\mathbf{x}; r')$ for some $r' \leq r$. You may wish to start by drawing some examples in $\mathbb{R}^2$.
   - Show that $U \subset \mathbb{R}^n$ is open in $(\mathbb{R}^n, d_1)$ if and only if it is an open in $(\mathbb{R}^n, d_2)$.
   - Show that a sequence $\{\mathbf{x}^k\} = \{x_1^k, \cdots x_n^k\}$ converges to some $\mathbf{x}^0$ in $(\mathbb{R}^n, d_1)$ if and only if it converges to $\mathbf{x}^0$ in $(\mathbb{R}^n, d_2)$. (Here superscripts denote place in the sequence and subscripts denote the Cartesian coordinates of a point.)
   - Show that $\{\mathbf{x}^k\}$ is a Cauchy sequence in $(\mathbb{R}^n, d_1)$ if and only if $\{\mathbf{x}_n\}$ is a Cauchy sequence in $(\mathbb{R}^n, d_2)$. Conclude that $(\mathbb{R}^n, d_2)$ is complete.